Rising Meson Trajectories*

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An integral equation for rising meson Regge trajectories is derived and solved for the ρ trajectory. The equation is a simple extension of a method previously described in which dispersion relations for $\alpha(s)$ and the reduced residue $b(s)$ are coupled by unitarity. In the present method, the trajectory dispersion relation is subtracted twice, which causes trajectories to rise $\sim s$. The solutions depend on two parameters and represent almost straight trajectories with imaginary parts.

I. INTRODUCTION AND REVIEW

THERE is considerable experimental[1] and theoretical[2] evidence that Regge trajectories rise to positive infinity with energy and fall to negative infinity with negative energy. In fact, if one takes recent indications seriously, one may conjecture that at least the ρ trajectory is roughly linear in s.[3] This also seems to agree well with several fits[4] to high-energy scattering data, where the ρ trajectory is exchanged.

If we consider the proposition that the Regge trajectory $\alpha_\rho(s)$ is a real-analytic function of s, with only a right-hand cut extending to infinity, then two reasonably obvious facts are worth mentioning:

(a) $\alpha(s)$ cannot be strictly linear in s, because in that case it could not have an imaginary part and the ρ meson itself would not have a width. Also, threshold properties of α would permit this only when $\alpha(s)=\frac{1}{2}$ at threshold.

(b) If we assume that the trajectory behaves as a power as $|s|\to\infty$, say, $\alpha(s)\sim|s|^n$, then real analyticity demands

$$\alpha(s)\to a(\cos\pi n)|s|^n-ia(\sin\pi n)|s|^n,\quad s\to+\infty+i\epsilon \tag{1a}$$

and

$$\alpha(s)\to a|s|^n,\quad s\to-\infty\,. \tag{1b}$$

In order for the trajectory to rise for positive s, we demand $a(\cos\pi n)>0$, and in order not to create infinitely many resonances with negative width, $a(\sin\pi n)<0$. This eliminates $0<n<\frac{1}{2}$ from our consideration (as it would $1<n<\frac{3}{2}$, etc.). Confining ourselves to $\frac{1}{2}<n\leq 1$, we see that we must have $a<0$ and if the trajectory will rise on the positive side it will fall on the negative side as $|s|\to\infty$.[5]

In this paper, using the ρ trajectory as an example, we combine statements which are often used separately, about analyticity of the trajectory and the residue function and unitarity, to derive an integral equation for $\alpha_\rho(s)$, which preserves its known properties and incorporates an imaginary part.

In previous papers a method was described[6] and applied to potential theory[7] which generated whole trajectories by combining the real analyticity of $\alpha(s)$ and of the reduced residue function $b(s)=\beta(s)q^{-2\alpha(s)}$. An integral equation for $\mathrm{Im}\alpha(s)$ was obtained by coupling the dispersion relations for α and b through unitarity. The residue β is not independent of the α's, but is a function of the partial-wave representation and of how many trajectories are to be simultaneously computed. In the crudest one-trajectory approximation, this yields $\beta=\mathrm{Im}\alpha/q$. This approximation comes from the simple application of unitarity to the expression

$$A(q,l)=\beta(q)/[l-\alpha(q)], \tag{2}$$

and neglects the contribution of other trajectories as well as the background integral.

Better representations which display the "full" contribution of each Regge pole, such as the Khuri and Cheng representations[8] and particularly modifications of these representations[6,9,10] which extract the high-energy part, greatly improved the convergence of these integral equations in terms of the number of trajectories which must be used in order to reproduce the exact results of the Schrödinger equations, so that only trajectories which actually reach the physical plane needed to be taken into consideration.[10,11] When $\mathrm{Im}\alpha(s)$ is much smaller than the spacing between trajectories

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[1] S. W. Karmanyos *et al.*, Phys. Rev. Letters **16**, 709 (1966); A. Citron *et al.*, Phys. Rev. **144**, 1101 (1966); W. Kienzle *et al.*, Phys. Letters **19**, 438 (1965); J. Sequinut *et al.*, *ibid.* **19**, 712 (1966); G. Chikovani *et al.*, *ibid.* **22**, 233 (1966); B. Levrat *et al.*, *ibid.* **22**, 714 (1966); M. Deutschmann *et al.*, *ibid.* **18**, 351 (1965); A. Forino *et al.*, *ibid.* **19**, 68 (1965); I. A. Vetlitsky *et al.*, *ibid.* **21**, 579 (1966); M. Focacci *et al.*, Phys. Rev. Letters **17**, 890 (1966).

[2] S. Mandelstam, University of California Radiation Laboratory Report No. UCRL-17250 (unpublished).

[3] D. Cline, Nuovo Cimento **45A**, 750 (1966); S. Minami, *ibid.* **46A**, 545 (1966); A. Ahmadzadeh, Nuovo Cimento **46A**, 415 (1966); P. C. O. Freund, Phys. Rev. **157**, 1412 (1967); D. G. Sutherland, CERN Report, 1967 (unpublished).

[4] F. Arbab and C. B. Chiu, Phys. Rev. **147**, 1045 (1966); C. B. Chiu, R. J. N. Phillips, and W. Rarita, *ibid.* **153**, 1485 (1967); G. Höhler, J. Baacke, H. Schaille, and P. Sondergger, Phys. Letters **20**, 79 (1966).

[5] Arguments to the contrary [N. N. Khuri, Phys. Rev. Letters **18**, 1094 (1967)] are based on single rising trajectories. While in the present approximations only a top trajectory will be computed, it is assumed that infinitely many will follow.

[6] S. C. Frautschi, P. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964); H. Cheng and D. Sharp, *ibid.* **132**, 1854 (1963).

[7] D. Hankins, P. Kaus, and C. J. Pearson, Phys. Rev. **137**, B1034 (1965); J. Blue, thesis, California Institute of Technology Report, 1966 (unpublished).

[8] N. N. Khuri, Phys. Rev. **130**, 429 (1963); H. Cheng, *ibid.* **144**, 1237 (1966).

[9] A. Ahmadzadeh, Phys. Rev. **133**, B1074 (1964).

[10] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **140**, B1595 (1965); **141**, 1513 (1966).

[11] W. J. Abbe and G. A. Gary, Phys. Rev. **160**, 1510 (1967).

($\sim$unity), then the detailed nature of the representation is much less important and the trajectories decouple to reasonably good approximation.

The whole scheme, however, had been based on trajectories which go to a constant as $s\to\infty$, which is correct for the usual class of potential-theory trajectories. We now want to calculate trajectories which have a power behavior as $s\to\infty$. In this case, the dispersion relations need more subtractions and the various representations[10] are suspect. In Sec. II we start from the beginning and derive a prototype equation, again using the simplest connection between residue and trajectory. With potential theory no longer a reliable guide, some of the boundary conditions are guesses and we can only hope that comparison with experiment will suggest better ones until we have a better understanding of the asymptotic properties suggested by rising trajectories and crossing.

II. INTEGRAL EQUATIONS

We consider the ρ trajectory $\alpha_\rho(s)$ and assume it obeys the dispersion relation

$$\alpha_\rho(\nu)=A+B(\nu-1)+\frac{\nu-1}{\pi}\int_0^\infty\frac{\mathrm{Im}\alpha_\rho(\nu')}{(\nu'-1)(\nu'-\nu)}d\nu',\quad(3)$$

where

$$\nu=(s-4m_\pi^2)/(m_\rho^2-4m_\pi^2).$$

For the reduced residue function, we want a function for which we can write an Omnes-type dispersion relation;

$$b(\nu)=g^2\exp\left[\frac{\nu-1}{\pi}\int_0^\infty\frac{\phi_b(\nu')}{(\nu'-1)(\nu'-\nu)}d\nu'\right]$$

$$b(\nu)=|b(\nu)|e^{i\phi_b(\nu)}.\quad(4)$$

The reduced residue function is proportional to the residue, which must vanish whenever the trajectory crosses a negative half-integer. In potential theory, this happens at the "indeterminacy points,"[12,13] which can be given as solutions to polynomials in s, the nth trajectory encountering $n-1$ such indeterminacy points. In order to avoid zeros in $b(\nu)$, it was possible to divide the residue $\beta_n(s)$ by $\prod^{n-1}(s-s_m)$, where the s_m are the $n-1$ indeterminacy points. However, for a trajectory which falls to negative infinity for negative ν, n becomes infinite. The function which naturally presents itself in this case instead of the inverse product of indeterminacy points is $\Gamma(\alpha(\nu)+\frac{3}{2})$ for the top trajectory. We now define

$$b(\nu)\equiv e^{c\nu}\Gamma(\alpha+\tfrac{3}{2})\beta(\nu)e^{-\alpha(\nu)\ln\nu}.\quad(5)$$

We again make a very simple ansatz for the connection between β and α;

$$\beta(\nu)=\frac{\mathrm{Im}\alpha(\nu)}{\nu^{1/2}}\left[\frac{4m_\pi^2}{m_\rho^2-4m_\pi^2}+\nu\right]^{1/2}e^{-d\nu},\quad(6)$$

where the exponential represents the effect of inelasticity in the $\pi\pi$ channel. The other part in (6) comes from applying elastic unitarity,

$$\frac{[A(s,l)-A^*(s,l^*)]}{2i}=\left[\frac{s-4m_\pi^2}{s}\right]^{1/2}A(s,l)A^*(s,l^*),$$

to the expression $A(s,l)=\beta/(l-\alpha)$ and going to the pole position $l=\alpha$. Equation (6) is definitely to be considered an approximation, while (3) and (4) are meant to be possibly wrong exact statements.

If we choose as a boundary condition that $b(\nu)$ should approach a constant as $s\to\infty$, we are now forced to set

$$(d-c)=B\ln B-B.\quad(7)$$

We then have, substituting (5)–(7) into (4),

$$\mathrm{Im}\alpha(\nu)=g^2\left[|\Gamma(\alpha(\nu)+\tfrac{3}{2})|\left(\frac{4m_\pi^2}{m_\rho^2-4m_\pi^2}+\nu\right)^{1/2}\right]^{-1}$$
$$\times\exp\left\{(B\ln B-B)\nu+[\tfrac{1}{2}+\mathrm{Re}\alpha(\nu)]\ln\nu+\frac{(\nu-1)}{\pi}P\int_0^\infty\frac{-\mathrm{Im}\alpha(\nu')\ln\nu'+\theta_\Gamma(\nu')}{(\nu'-\nu)(\nu'-1)}d\nu'\right\},\quad(8)$$

where

$$\Gamma(\alpha+\tfrac{3}{2})=|\Gamma(\alpha+\tfrac{3}{2})|e^{i\theta_\Gamma(\nu')}.$$

Substituting (3), the dispersion relation for $\alpha(\nu)$, into (8) gives the integral equation

$$\mathrm{Im}\alpha(\nu)=G^2\nu^{3/2}\left[|\Gamma(\alpha(\nu)+\tfrac{3}{2})|\left(\frac{4m_\pi^2}{m_\rho^2-4m_\pi^2}+\nu\right)^{1/2}\right]^{-1}$$
$$\times\exp\left\{(\nu-1)\left[B(\ln(B\nu)-1)+\frac{P}{\pi}\int_0^\infty\frac{\mathrm{Im}\alpha(\nu')\ln(\nu/\nu')+\theta_\Gamma(\nu')}{(\nu'-\nu)(\nu'-1)}d\nu'\right]\right\},\quad(9)$$

where

$$G^2=g^2e^{(B\ln B-B)}.$$

Since we are dealing with the ρ trajectory, we have set the subtraction constant A, which is the real part of the trajectory at the ρ mass, equal to unity.

The strange combination of factors, $B\ln B-B$, comes about because of the asymptotic property of $\Gamma(Z)$;

$$\Gamma(Z)\to(2\pi)^{1/2}e^{(Z\ln Z-Z)-\frac{1}{2}\ln Z}[1+O(Z^{-1})].\quad(10)$$

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[12] See, e.g., P. Kaus, Nuovo Cimento 29, 598 (1963).

[13] However, such a calculation must be approached with caution, because of threshold difficulties when $\alpha(\nu=0)<-\frac{1}{2}$. See, e.g., Ref. 7, Appendix.

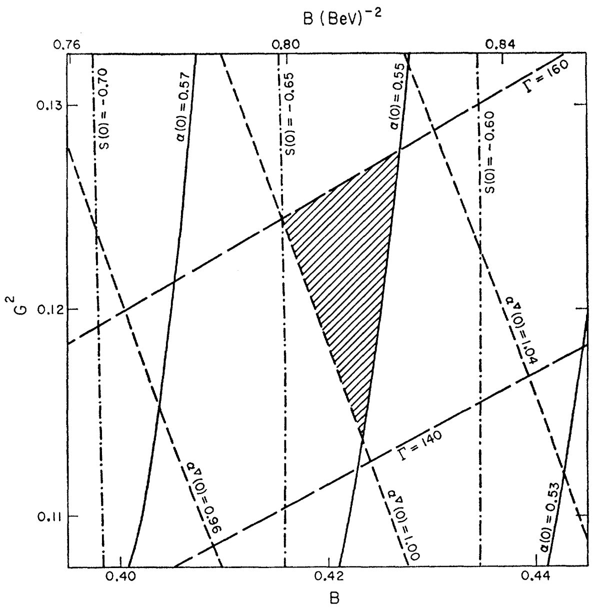

FIG. 1. Properties of solutions of Eqs. (11) as function of parameters G^2 and B. Shown are lines of constant ρ width, Γ, in MeV; position of α at $s=0$, $\alpha(0)$; slope at $s=0$, $\alpha'(0)$, in BeV^{-2} and s for $\alpha=0$, $s(0)$, in BeV^2. The parameter B (the asymptotic slope) labels the horizontal axis in dimensionless units at the bottom and in BeV^{-2} at the top. The shaded area represents acceptable parameters.

III. NUMERICAL RESULTS

The system of equations, which was solved for the ρ trajectory, is given by

$$\operatorname{Re}\alpha(\nu)=1+(\nu-1)\left[B+\frac{P}{\pi}\int_0^\infty \frac{\operatorname{Im}\alpha(\nu')}{(\nu'-1)(\nu'-\nu)}d\nu'\right],$$

$$\operatorname{Im}\alpha(\nu)=G^2\nu^{3/2}\left[\left|\Gamma(\alpha(\nu)+\tfrac{3}{2})\right|\left(\frac{4m_\pi^2}{m_\rho^2-4m_\pi^2}+\nu\right)^{1/2}\right]^{-1}$$

$$\times\exp\left\{(\nu-1)\left[B(\ln(B\nu)-1)\right.\right.$$

$$\left.\left.+\frac{P}{\pi}\int_0^\infty\frac{\operatorname{Im}\alpha(\nu')\ln(\nu/\nu')+\theta\,\Gamma(\nu')}{(\nu'-\nu)(\nu'-1)}d\nu'\right]\right\}. \quad (11)$$

This leaves two parameters B and G^2, where B represents the ultimate slope of the trajectory and G^2 is related to $\operatorname{Im}\alpha(\nu)$ at $\nu=1$ (the ρ mass squared), and hence the width of the ρ.

These Eqs. (11) were solved numerically. They converged readily on iteration, starting with $\operatorname{Im}\alpha(\nu)=0$ as the first input.

In Fig. 1 some of the main properties of the solutions are represented as functions of the input parameters G^2 and B. All solutions give a ρ meson with mass $m_\rho=770$ MeV. The figure shows contours of constant width, the intercept $\alpha(0)\equiv\alpha(s=0)$, and $\alpha'(0)\equiv d\alpha/ds|_{s=0}$. The asymptotic dimensionless slope B is also marked in BeV^{-2} for comparison with $\alpha'(0)$ the slope at $s=0$. Some phenomenological fits[14] for $\alpha(0)$ and $\alpha'(0)$ are marked on the figure. This should not be taken too seriously, because these are straight-line fits and our solution has curvature, though small, from the dispersion integral.

Typical solutions are shown in some detail in Figs. 2(a), 2(b), and 3. It is seen that because of the subtracted dispersion relation for $\alpha(s)$, the deviations from a straight line are small, even though there is a reasonably large imaginary part. Nevertheless, in the region $-1<s<1$ BeV^2 it is not negligible. This is most probably not a feature of the particular boundary conditions chosen for this integral equation (11), but is just due to the width of the ρ in any system. This statement should be approached with caution, as will be pointed out in Sec. IV.

Finally, the reduced residue function is shown in Fig. 4, for a typical case, ($G^2=0.1233$, $B=0.4213$), the same as in Fig. 2. Note that $b(s)$ is defined by (5) and

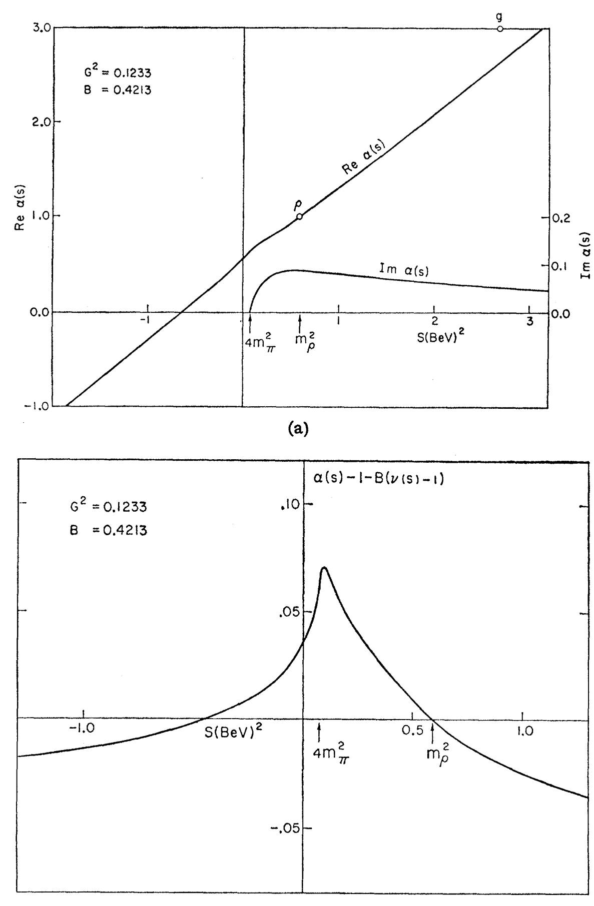

FIG. 2. (a) The ρ trajectory corresponding to the solution of (11) with $G^2=0.1233$ and $B=0.4213$ (≈ 0.81 BeV^{-2}). (b) The difference between the solution in 2(a) and a straight-line trajectory given by $\alpha=1+B(\nu-1)$,

[14] See Refs. 3 and 4.

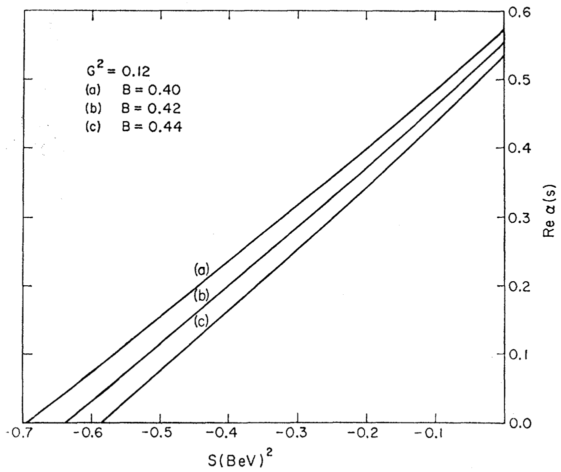

FIG. 3. Solutions corresponding to $G^2=0.12$ and several values of B for negative s. Note that they are not straight lines.

may or may not agree with anybody else's definition of "reduced residue."

IV. DISCUSSION

The integral equation (11) represents the simplest extension of the method described in Ref. 6, which worked well in the potential-theory framework.[7] The boundary condition $\mathrm{Re}\alpha(s)\rightarrow s$ is not unique and until we better understand all the consequences of rising trajectories, we cannot settle this question. Recently, Khuri[5] proved that if a single trajectory rises, $\mathrm{Re}\alpha(s)\rightarrow s^n$, then n must be smaller than $\frac{1}{2}$. However, from (1a) we see that this would produce ghosts. The way out of this difficulty probably comes from the need for more trajectories (daughters), all rising as the same power.

In the present model, the effect of the imaginary part on the real part of the trajectory is never large. Near threshold the imaginary part, and hence the curvature, of the trajectory depends mostly on threshold conditions, given by unitarity and the position and width of the ρ meson, namely, $\mathrm{Im}\alpha(s)\approx 0.1$ at the ρ mass. More important, perhaps, is the effect of the real part on the imaginary part. It is seen from Fig. 1, for instance, that to reach the favored parameters for straight-line fits when the ρ is exchanged,[4] $\alpha(0)\approx 0.58$ and $\alpha'(0)\approx 1.0$, would imply a width for the ρ larger than 180 MeV.

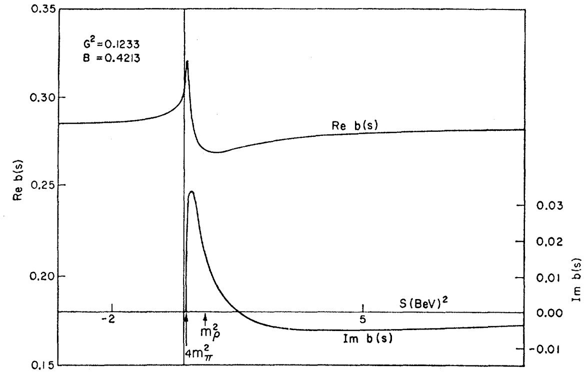

FIG. 4. The reduced residue function $b(s)$ for the solution of Fig. 2.

These specific conclusions may change somewhat with different coupling equations (6) or boundary conditions at infinity, which are implied by (4) and (5). There has been no attempt made here to fit higher recurrences[1] of the ρ, but it is easy to see from Fig. 1 that asymptotic slopes B which fit the negative s region, i.e., around 0.82, are too small to give a good ρ meson ($\alpha=3, s\approx 2.7$) recurrence.

In connection with other boundary conditions it should be mentioned that in the present model $\mathrm{Im}\alpha(s)\rightarrow \sim s^{-1}$. This is consistent with (1), implying a term $\mathrm{Re}\alpha(s)\rightarrow \sim s^{-1}\ln s$ which will be dominated by the term proportional to s. It is perhaps more reasonable to assume $\mathrm{Im}\alpha(s)\rightarrow \sim c$ or $\sim s^{1/2}$.

In order to achieve this, the Omnes equation (4) can be written for a function $\bar{b}(\nu)$, which is to be defined as going to a constant when $\mathrm{Im}\alpha(\nu)$ has the new behavior. Two possibilities present themselves; one is

$$\bar{b}(\nu)=b(\nu)/(\nu-\nu_0). \tag{12}$$

In this case (9) will be multiplied by $\nu-\nu_0$ on the right side. The new parameter ν_0 represents the somewhat arbitrary point at which $\beta(\nu)$ vanishes.[15] Another alternative is to write the dispersion relation (4) for the function

$$\bar{b}=b(\nu)/[\alpha(\nu)+N]. \tag{13}$$

In both cases (12) and (13), $b(\nu)$ is defined by (5)–(7) and in both cases $\mathrm{Im}\alpha(s)\rightarrow c$ after (12) and (13) are used to get the integral equation equivalent to (9). The second possibility [Eq. (13)] leads to

$$\mathrm{Im}\alpha(\nu)=G^2\left|(\alpha(\nu)+N)\Gamma^{-1}(\alpha(\nu)+\tfrac{3}{2})\right|\left(\frac{4m_\pi^2}{m_\rho^2-4m_\pi^2}+\nu\right)^{-1/2}$$
$$\times\exp(\nu-1)\left[B(\ln(B\nu)-1)+\frac{P}{\pi}\int_0^\infty\frac{\mathrm{Im}\alpha(\nu\,)\ln(\nu/\nu')-\tan^{-1}[\mathrm{Im}\alpha(\nu')/\mathrm{Re}\alpha(\nu')]+\theta_\Gamma(\nu')}{(\nu'-\nu)(\nu'-1)}d\nu'\right]. \tag{14}$$

[15] In this connection see the discussion on number of zeros in $b(\nu)$ in Ref. 6.

The significance of the parameter N here is that it is a "residue-killing" factor when the trajectory passes through $\alpha(\nu)=-N$. For large $|\nu_0|$ in (12) or large N in (13), the equations will change very little. But if the zero is imposed near the threshold ($\nu\approx 0$) or zero-mass ($s\approx 0$) region, the effect would be considerable.

If we wish to calculate daughter[16] trajectories, then for the Nth daughter, the residue function (5) will have $\Gamma(\alpha+\frac{3}{2})$ replaced by $\Gamma(\alpha+\frac{3}{2}+N)$ and the subtraction constant A in (3) adjusted to give a two-unit spacing at $s=0$, i.e., $A_N\approx A-2N$.[13] It is clear that with the inclusion of other trajectories, the coupling of the top trajectories to each other must be taken more seriously. We do not know how to do this, however, until we have a reliable partial-wave representation for the case of (possibly infinitely many) rising trajectories.

Finally, a word about models. From potential theory we learned that with the usual kind of potentials, rising trajectories are very hard to come by in a single-channel problem. However, when we have high-spin resonances at high energies, many decay channels are available. Higher channels exert an attractive force on bound states and resonances of lower channels especially near their threshold, and as more channels open up this may carry trajectories higher indefinitely. Chew and Jones[17] have proposed a model along this line, where a trajectory is dominated by the highest-spin, lowest-orbital-angular-momentum channels, whose thresholds lie in that region.

Another way to think about rising trajectories is to consider a two-particle channel, the physical channel, with a usual potential, coupled into a channel which in the absence of coupling is something like a harmonic oscillator. The physical channel, unitary above its threshold, will contain all the Regge poles of the harmonic oscillator, which has no threshold, but in the absence of coupling has Regge trajectories of the form[18]

$$\alpha_N(\nu)=A-\tfrac{3}{2}+B\nu-2N\,,\quad N=0,1,2\cdots,\qquad(15)$$

[16] G. Domokos and P. Suranyi, Nucl. Phys. **54**, 529 (1964); D. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1569 (1967).

[17] Private communication: see also S. Y. Chu and C. J. Tan, University of California Radiation Laboratory Report No. UCRL-17511, 1967 (unpublished).

[18] To get the closest thing to an S matrix for a harmonic oscillator from that for the Coulomb potential, let $x=r^2$ and $\psi(x)=r^{1/2}\varphi(r)$. D. I. Fivel, Phys. Rev. **142**, 1219 (1966).

where

$$V(r)=(1/B)(r^2/4B-A)\,.$$

The residues of the corresponding horribly nonunitary "S matrix" are given by

$$R_N(\nu)=(-1)^N/2N!\,\Gamma(\alpha_N(\nu)+\tfrac{3}{2}+N)\,.\qquad(16)$$

The indeterminacy points of (5) are displayed in (16). Our Eqs. (11) can then be thought of as what happens when such an infinite-energy-threshold channel is coupled to the finite threshold π-π channel. It is amusing to think of the harmonic oscillator channel as a quark-antiquark channel, the constant g^2 in (4) representing the $\pi\pi\rightarrow q\bar{q}$ coupling. Alternatively, it may be considered as representing infinitely many uncounted channels. Even more vaguely, the two statements may have the same meaning.

Refinement of the integral equations must wait for better representations in which the high-energy part is again subtracted out smoothly[6] and without double counting. When we have such a representation, we can improve the coupling formula (6) and consider many trajectories as well as the question of crossing symmetry[10] and bootstrap.[19]

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[19] *Note added in manuscript.* S. Mandelstam has also proposed equations of the type (3)–(6) in a recent manuscript [S. Mandelstam, Phys. Rev. **166**, 1539 (1968)]. He discusses the application to bootstrap, by solving the equations in the narrow-resonance approximation and introducing crossing by means of the generalized superconvergence relation of Igi and of Horn and Schmid.